

Multipartite pure-state entanglement and the generalized GHZ states

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We show that not all 4-party pure states are GHZ reducible (i.e., can be generated reversibly from a combination of 2-, 3- and 4-party maximally entangled states by local quantum operations and classical communication asymptotically) through an example, we also present some properties of the relative entropy of entanglement for those 3-party pure states that are GHZ reducible, and then we relate these properties to the additivity of the relative entropy of entanglement.

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I. INTRODUCTION

Ever since it was first noted by Einstein-Podolsky-Rosen (EPR) [1] and Schrödinger [2], entanglement has played an important role in quantum information theory. Quantum entanglement provides strong tests of quantum nonlocality [3,4], and it is also a useful resource for various kinds of quantum information processing, including teleportation [5,6], cryptographic key distribution [7], quantum error correction [8] and quantum computation [9].

Now, one of the key open questions in quantum information theory is how many fundamentally different types of quantum entanglement there are. It was known [10,11] that asymptotically there is only one kind of entanglement for bipartite pure states, any pure entangled state of two parties (Alice and Bob) may be reversibly transformed into EPR states by local quantum operations and classical communication (LOCC) asymptotically.

For multipartite pure states, it is more difficult to understand the types of entanglement. It was not known whether the EPR states are the only type of entanglement until the recent work of Bennett-Popescu-Rohrlich-Smolín-Thapliyal (BPRST) [12], which shows that the 4-party GHZ state can not be reversibly transformed into EPR states by LOCC asymptotically. Furthermore Linden-Popescu-Schumacher-Westmoreland (LPSW) [13] had shown that the n -party GHZ state cannot be reversibly transformed into any combination of k -party entangled pure states for all $k < n$. This means that the generalized n -party GHZ state

$$|GHZ_n\rangle_{ABC\dots} \equiv \frac{1}{\sqrt{2}}(|0^{\otimes n}\rangle + |1^{\otimes n}\rangle) \quad (1)$$

represents a different type of entanglement with respect

to the k -party GHZ state (for all $k \neq n$)¹.

A natural question arises, are the generalized GHZ states the only types of entanglement? Thapliyal [14] had shown that any multi-separable pure state is Schmidt decomposable, thus a m -party separable pure state (a state contains no entanglement of k -party for all $k < m$) can be reversibly transformed into the m -party GHZ state, this result supports (but does not prove) the hypothesis that the generalized GHZ states are the only types of entanglement with the k -party GHZ state representing "essential" k -party entanglement.

In this note, we show that the generalized GHZ states are not the only types of entanglement through an example of 4-party pure state. We also present some properties of the relative entropy of entanglement for those 3-party pure states that can be generated reversibly from 2- and 3-party GHZ states, and then we use these properties to analyze the additivity of the relative entropy of entanglement.

Before going to the results, we state some terminology more clearly. Two m -party pure states $|\psi\rangle$ and $|\varphi\rangle$ are LOCCa equivalent if and only if [12]

$$\begin{aligned} \forall \delta > 0, \epsilon > 0 \exists_{n_1, n_2, n_3, n_4, L, L'} \text{ so that} \\ |(n_1/n_2) - 1| < \delta, \\ |(n_3/n_4) - 1| < \delta, \text{ and} \\ F(L(|\psi\rangle^{\otimes n_1}), |\varphi\rangle^{\otimes n_2}) \geq 1 - \epsilon \\ F(L'(|\psi\rangle^{\otimes n_3}), |\varphi\rangle^{\otimes n_4}) \geq 1 - \epsilon \end{aligned} \quad (2)$$

where L and L' are local quantum operations assisted by classical communication, and

$$F(|\Phi\rangle, |\Psi\rangle) \equiv |\langle \Phi | \Psi \rangle|^2 \quad (3)$$

¹A particular n -party GHZ state is chosen to represent all the n -party GHZ states since they are related by local unitary transformations.

is the fidelity of $|\Psi\rangle$ relative to $|\Phi\rangle$. Condition (2) means that, in the limit of large n , n copies of $|\psi\rangle$ can be transformed into almost the same number of copies of $|\varphi\rangle$ by LOCC with high fidelity, and vice versa. The LOCCa equivalence of the two m -party pure states $|\psi\rangle$ and $|\varphi\rangle$ is denoted as

$$|\psi\rangle \stackrel{LOCCa}{\rightleftharpoons} |\varphi\rangle \quad (4)$$

We say that a m -party pure state $|\psi\rangle$ is GHZ reducible, if and only if the state $|\psi\rangle$ is LOCCa equivalent to a combination of 2-, 3-, \dots , m -party generalized GHZ states. For example, since any bipartite pure state $|\psi\rangle_{AB}$ is GHZ reducible [12,14], we can write

$$|\psi\rangle_{AB} \stackrel{LOCCa}{\rightleftharpoons} |EPR\rangle_{AB}^{\otimes E(|\psi\rangle_{AB})} \quad (5)$$

where $E(|\psi\rangle_{AB})$ is the unique measure of entanglement for bipartite pure states, it is equal to the entropy of the reduced density matrix of either Alice or Bob, as well as the entanglement of formation [15], entanglement of distillation [10,16] and the relative entropy of entanglement [17]. A 3-party GHZ reducible pure state $|\psi\rangle_{ABC}$ can be written as

$$|\psi\rangle_{ABC} \stackrel{LOCCa}{\rightleftharpoons} |EPR\rangle_{AB}^{\otimes E_2(AB)} \otimes |EPR\rangle_{AC}^{\otimes E_2(AC)} \otimes |EPR\rangle_{BC}^{\otimes E_2(BC)} \otimes |GHZ\rangle_{ABC}^{\otimes E_3(ABC)} \quad (6)$$

i.e., in the limit of large n , with high fidelity, n copies of the state $|\psi\rangle_{ABC}$ can be transformed reversibly by LOCC into $n \cdot E_2(AB)$ copies of the state $|EPR\rangle_{AB}$ held by Alice and Bob, $n \cdot E_2(AC)$ copies of $|EPR\rangle_{AC}$ held by Alice and Claire, $n \cdot E_2(BC)$ copies of $|EPR\rangle_{BC}$ held by Bob and Claire, and $n \cdot E_3(ABC)$ copies of $|GHZ\rangle_{ABC}$ held by Alice, Bob and Claire. The GHZ reducible multipartite pure states can be written in similar forms.

Let \mathcal{C} denotes a set of pure states, if each of the m -party pure states is LOCCa equivalent to a certain combination of the states in \mathcal{C} , then we say that \mathcal{C} is a reversible entanglement generating set (REGS) [12] for m -party pure states. A minimal reversible entanglement

generating set (MREGS) for m -party pure states is a REGS of minimal cardinality. It is obvious that the set $\mathcal{C}_2 = \{|EPR\rangle_{AB}\}$ is a MREGS for bipartite pure states. The question that whether the set of 2-, 3-, \dots , m -party generalized GHZ states is a MREGS for m -party pure states is, in fact, equivalent to the question that whether all m -party pure states are GHZ reducible.

We now state BPRST's lemma about the LOCC equivalence.

BPRST's lemma: If two m -party quantum states $|\Psi\rangle$ and $|\Phi\rangle$ are LOCCa equivalent, then they must be isotropic, i.e.,

$$S_X(|\Psi\rangle) = S_X(|\Phi\rangle) \quad (7)$$

where $S_X(|\Psi\rangle) = -\text{tr}\{\rho_X(|\Psi\rangle) \log_2 \rho_X(|\Psi\rangle)\}$ with $\rho_X(|\Psi\rangle) \equiv \text{tr}_{\overline{X}}(|\Psi\rangle\langle\Psi|)$, and X denotes a nontrivial subset of the parties (say Alice, Bob, Claire, Daniel, et al.), \overline{X} denotes the set of the remaining parties.

This lemma is a consequence of the fact that average partial entropy S_X cannot increase under LOCC, details of proof can be found in ref. [16,12].

II. THE SET OF GENERALIZED GHZ STATES IS NOT A MREGS

Now we show that the generalized GHZ states are not the only types of entanglement by proving that the set of 2-, 3-, 4-party GHZ states is not a MREGS for 4-party pure states, or in another word, not all 4-party pure states are GHZ reducible.

Proposition 1: The set of 2-, 3-, 4-party GHZ states is not a MREGS for 4-party pure states.

Before the proof, let us first give a property of all the GHZ reducible 4-party pure states. Suppose the 4-party pure state $|\Psi\rangle_{ABCD}$ is GHZ reducible, i.e.,

$$\begin{aligned} |\Psi\rangle_{ABCD} \stackrel{LOCCa}{\rightleftharpoons} & |EPR\rangle_{AB}^{\otimes E_2(AB)} \otimes |EPR\rangle_{AC}^{\otimes E_2(AC)} \otimes |EPR\rangle_{AD}^{\otimes E_2(AD)} \otimes |EPR\rangle_{BC}^{\otimes E_2(BC)} \\ & \otimes |EPR\rangle_{BD}^{\otimes E_2(BD)} \otimes |EPR\rangle_{CD}^{\otimes E_2(CD)} \otimes |GHZ\rangle_{ABC}^{\otimes E_3(ABC)} \otimes |GHZ\rangle_{ABD}^{\otimes E_3(ABD)} \\ & \otimes |GHZ\rangle_{ACD}^{\otimes E_3(ACD)} \otimes |GHZ\rangle_{BCD}^{\otimes E_3(BCD)} \otimes |GHZ_4\rangle_{ABCD}^{\otimes E_4(ABCD)} \end{aligned} \quad (8)$$

From BPRST's lemma and the additivity of the von Neumann entropy, we have

$$\begin{aligned} S(\rho_A) &= E_2(AB) + E_2(AC) + E_2(AD) + E_3(ABC) + E_3(ABD) + E_3(ACD) + E_4(ABCD) \\ S(\rho_B) &= E_2(AB) + E_2(BC) + E_2(BD) + E_3(ABC) + E_3(ABD) + E_3(BCD) + E_4(ABCD) \\ S(\rho_C) &= E_2(AC) + E_2(BC) + E_2(CD) + E_3(ABC) + E_3(ACD) + E_3(BCD) + E_4(ABCD) \\ S(\rho_D) &= E_2(AD) + E_2(BD) + E_2(CD) + E_3(ABD) + E_3(ACD) + E_3(BCD) + E_4(ABCD) \end{aligned} \quad (9)$$

and

$$\begin{aligned} S(\rho_{AB}) &= E_2(AC) + E_2(AD) + E_2(BC) + E_2(BD) + E_3(ABC) \\ &\quad + E_3(ABD) + E_3(ACD) + E_3(BCD) + E_4(ABCD) \\ S(\rho_{AC}) &= E_2(AB) + E_2(AD) + E_2(BC) + E_2(CD) + E_3(ABC) \\ &\quad + E_3(ABD) + E_3(ACD) + E_3(BCD) + E_4(ABCD) \\ S(\rho_{AD}) &= E_2(AB) + E_2(AC) + E_2(BD) + E_2(CD) + E_3(ABC) \\ &\quad + E_3(ABD) + E_3(ACD) + E_3(BCD) + E_4(ABCD) \end{aligned} \quad (10)$$

From Eqs. (9) it follows that

$$\sum_{i \in \{A, B, C, D\}} S(\rho_i) = 2 \cdot E_{2t} + 3 \cdot E_{3t} + 4 \cdot E_4 \quad (11)$$

with E_{2t} (E_{3t} , E_4) representing the "total" 2- (3-, 4-) party entanglement, which is defined by

$$\begin{aligned} E_{2t} &= E_2(AB) + E_2(AC) + E_2(AD) \\ &\quad + E_2(BC) + E_2(BD) + E_2(CD) \\ E_{3t} &= E_3(ABC) + E_3(ABD) \\ &\quad + E_3(ACD) + E_3(BCD) \\ E_4 &= E_4(ABCD) \end{aligned} \quad (12)$$

And from Eqs. (10), there is

$$S(\rho_{AB}) + S(\rho_{AC}) + S(\rho_{AD}) = 2 \cdot E_{2t} + 3 \cdot E_{3t} + 3 \cdot E_4 \quad (13)$$

It follows from eq. (11) and (13) that

$$E_4 = \sum_{i \in \{A, B, C, D\}} S(\rho_i) - \{S(\rho_{AB}) + S(\rho_{AC}) + S(\rho_{AD})\} \quad (14)$$

This is the amount of "essential" 4-party entanglement contained in the state $|\Psi\rangle_{ABCD}$, therefore it must be non-negative, i.e.,

$$\sum_{i \in \{A, B, C, D\}} S(\rho_i) - \{S(\rho_{AB}) + S(\rho_{AC}) + S(\rho_{AD})\} \geq 0 \quad (15)$$

Eq. (15) is a property of all the GHZ reducible 4-party pure states. Similar results for m -party GHZ reducible pure states can follow from the same argument.

Now let us take the state

$$|\psi\rangle_{ABCD} = \frac{1}{2} \{|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle\} \quad (16)$$

as an example. It's obvious that

$$\begin{aligned} S(\rho_A) &= S(\rho_B) = S(\rho_C) = S(\rho_D) = 1 \\ S(\rho_{AB}) &= S(\rho_{AC}) = 2 \\ S(\rho_{AD}) &= 1 \end{aligned} \quad (17)$$

therefore

$$E_4 = 4 \times 1 - (2 + 2 + 1) = -1 \quad (18)$$

This contradicts eq. (15). Thus we have shown that not all 4-party pure states are GHZ reducible, so the set of 2-, 3-, 4-party GHZ states is not a MREGS for 4-party pure states. This completes the proof of proposition 1.

Proposition 1 shows that the set of 11 generalized GHZ states in eq. (8) is not enough for a MREGS, i.e., the number of members in a MREGS for 4-party pure states must be greater than 11.

III. GHZ REDUCIBLE TRIPARTITE PURE STATES

It was known that any bipartite pure state is GHZ reducible, and from proposition 1 we know that not all 4-party pure states are GHZ reducible. It is natural to ask whether all tripartite pure states are GHZ reducible. The answer of this question is not found yet, however we give some properties of the GHZ reducible tripartite pure states.

Let us first recall the definitions of the relative entropy of entanglement and Rains' bound of entanglement. Let the systems A and B be in a joint state ρ_{AB} , the relative entropy of entanglement $E_r(A, B)$ is defined by [17]

$$\begin{aligned} E_r(A, B) &\equiv E_r(\rho_{AB}) \\ &\equiv \min_{\sigma \in \tilde{D}} tr_{AB} \{\rho_{AB} (\log_2 \rho_{AB} - \log_2 \sigma)\} \end{aligned} \quad (19)$$

where \tilde{D} is the set of all disentangled states of the two systems A and B. Let \tilde{P} be the set of all bipartite states that have positive partial transposes (PPT), similarly Rains' bound of entanglement B_Γ is defined by [19]

$$B_\Gamma(\rho_{AB}) \equiv \min_{\sigma \in \tilde{P}} tr_{AB} \{\rho_{AB} (\log_2 \rho_{AB} - \log_2 \sigma)\} \quad (20)$$

It is obvious that $B_\Gamma(\rho_{AB}) \leq E_r(\rho_{AB})$ since any separable state is a PPT state [22]. Now we give the following proposition about the relative entropy of entanglement for a special kind of tripartite pure states.

Proposition 2: For the 3-party pure state

$$|\Phi\rangle_{ABC} = |\psi\rangle_{AB}^{\otimes m} \otimes |\varphi\rangle_{AC}^{\otimes n} \otimes |\phi\rangle_{BC}^{\otimes l} \otimes |\Theta\rangle_{ABC}^{\otimes k} \quad (21)$$

where the state $|\Theta\rangle_{ABC}$ is Schmidt decomposable, (i.e., $|\Theta\rangle_{ABC} = \sum_i \sqrt{p_i} |i\rangle_A |i\rangle_B |i\rangle_C$) there is

$$\begin{aligned} E_r(A, B) &= m \cdot E_r(|\psi\rangle_{AB}) \\ E_r(A, C) &= n \cdot E_r(|\varphi\rangle_{AC}) \\ E_r(B, C) &= l \cdot E_r(|\phi\rangle_{BC}) \end{aligned} \quad (22)$$

If the relative entropy of entanglement is additive, proposition 2 is obviously true. However, the additivity of the relative entropy of entanglement has not been proved yet (maybe it is not provable at all), so this proposition should be proved. A proof of this proposition can be found in Appendix A, here we prove this proposition by proving the following lemma.

Lemma 1: For a bipartite pure state ρ and a bipartite separable state ρ' , there is $E_r(\rho \otimes \rho') = E_r(\rho)$.

Proof. On one hand, it is obvious that [18]

$$E_r(\rho \otimes \rho') \leq E_r(\rho). \quad (23)$$

On the other hand, as a property of B_Γ , there is [19]

$$B_\Gamma(\rho \otimes \rho') = B_\Gamma(\rho) = E_r(\rho) \quad (24)$$

i.e.,

$$E_r(\rho \otimes \rho') \geq B_r(\rho \otimes \rho') = E_r(\rho) \quad (25)$$

Thus lemma 1 follows from eqs. (23) and (25). From lemma 1, and the additivity of the relative entropy of entanglement for pure states, proposition 2 can easily be proved.

LPSW's lemma: If two 3-party (Alice,Bob,Claire) quantum states $|\Psi\rangle$ and $|\Phi\rangle$ are LOCCa equivalent, then each of the relative entropies of entanglement of $|\Psi\rangle$ is equal to the corresponding one of $|\Phi\rangle$, i.e.,

$$\begin{aligned} E_r^{|\Psi\rangle}(A, B) &= E_r^{|\Phi\rangle}(A, B) \\ E_r^{|\Psi\rangle}(A, C) &= E_r^{|\Phi\rangle}(A, C) \\ E_r^{|\Psi\rangle}(B, C) &= E_r^{|\Phi\rangle}(B, C) \end{aligned} \quad (26)$$

This lemma follows from LPSW's inequality that for any LOCC protocol, the average increase in $E_r(B, C)$ is no greater than the average decrease in the entanglement between Alice and the joint Bob-Claire system. Detailed discussion can be found in ref. [13]. By this lemma, LPSW had made quantitative statements about tripartite entanglement, they notice that there are relations between the one-party entropies and relative entropies. Here we look more carefully into this issue and extract the relations of the entropies.

Proposition 3: If tripartite pure state $|\Psi\rangle_{ABC}$ is GHZ reducible, then there must be

$$\begin{aligned} S(\rho_A) + E_r(B, C) &= S(\rho_B) + E_r(A, C) \\ &= S(\rho_C) + E_r(A, B) \end{aligned} \quad (27)$$

and

$$\begin{aligned} S(\rho_A) &\geq E_r(A, B) + E_r(A, C) \\ S(\rho_B) &\geq E_r(A, B) + E_r(B, C) \\ S(\rho_C) &\geq E_r(A, C) + E_r(B, C) \end{aligned} \quad (28)$$

where $S(\rho_A)$ is the von Neumann entropy of the reduced density matrix of system A, and $E_r(A, B)$ is the relative entropy of entanglement of the systems A+B.

Proof. Since $|\Psi\rangle_{ABC}$ is GHZ reducible, i.e.,

$$\begin{aligned} |\Psi\rangle_{ABC} &\stackrel{LOCCa}{\rightleftharpoons} |\Phi\rangle_{ABC} \equiv |EPR\rangle_{AB}^{\otimes E_2(AB)} \\ &\otimes |EPR\rangle_{AC}^{\otimes E_2(AC)} \otimes |EPR\rangle_{BC}^{\otimes E_2(BC)} \otimes |GHZ\rangle_{ABC}^{\otimes E_3(ABC)} \end{aligned} \quad (29)$$

From LPSW's lemma and proposition 2, we have

$$\begin{aligned} E_2(AB) &= E_r(A, B) \\ E_2(AC) &= E_r(A, C) \\ E_2(BC) &= E_r(B, C) \end{aligned} \quad (30)$$

From eq. (30) and the additivity of the von Neumann entropy, it follows that

$$\begin{aligned} S(\rho_A) &= E_r(A, B) + E_r(A, C) + E_3(ABC) \\ S(\rho_B) &= E_r(A, B) + E_r(B, C) + E_3(ABC) \\ S(\rho_C) &= E_r(A, C) + E_r(B, C) + E_3(ABC) \end{aligned} \quad (31)$$

Since $E_3(ABC) \geq 0$, proposition 3 follows from eq. (31).

Eqs. (30) and (31) are also obtained in ref. [13]. If we suppose that the relative entropy of entanglement is additive, then eqs. (30), (31) and proposition 3 are obvious results, however, here we have given a proof of these results without the assumption of additivity.

We do not know whether conditions (27) and (28) are satisfied by all tripartite pure states, but it can be shown that eq. (27) is satisfied for the following case.

Proposition 4: For the tripartite pure state $|\Psi\rangle_{ABC}$, there are 3 reduced density matrixes of two parties, ρ_{AB} , ρ_{AC} and ρ_{BC} , if at least two of them are separable states, then eq. (27) is satisfied.

Proof of proposition 4 is left to Appendix B.

IV. REDUCIBILITY OF TRIPARTITE PURE STATES AND ADDITIVITY OF THE RELATIVE ENTROPY OF ENTANGLEMENT

Let Alice (Bob) hold systems 1 and 3 (2 and 4), ρ_{12} (ρ_{34}) be the joint state of the systems 1 and 2 (3 and 4), and let the systems 1+2 be uncorrelated with the systems 3+4, i.e., the overall state of the systems 1+2+3+4 can be written as

$$\rho_{AB} = \rho_{12} \otimes \rho_{34} \quad (32)$$

We would like to have the additivity

$$E_r(A, B) = E_r(\rho_{AB}) = E_r(\rho_{12}) + E_r(\rho_{34}) \quad (33)$$

as an important property desired from a measure of entanglement [18,12]. The additivity has been proved for the case that both ρ_{12} and ρ_{34} are pure states [18], for more general cases, it remains a conjecture.

Proposition 5: The relative entropy of entanglement is additive if each of the two uncorrelated states (i.e., the above states ρ_{12} and ρ_{34}) can be purified into a GHZ reducible tripartite pure state.

Proposition 5 says that, if there are two GHZ reducible tripartite pure states $|\psi\rangle_{125}$ and $|\varphi\rangle_{346}$ such that

$$\begin{aligned} \rho_{12} &= \text{tr}_5 \{ |\psi\rangle_{125} \langle \psi| \} \\ \rho_{34} &= \text{tr}_6 \{ |\varphi\rangle_{346} \langle \varphi| \} \end{aligned} \quad (34)$$

then eq. (33) is satisfied. This proposition follows directly from proposition 2. And we give the following proposition as a corollary.

Proposition 6: If all the tripartite pure states are GHZ reducible, then the relative entropy of entanglement is generally additive.

In another word, if we can find a counter-example for the additivity of the relative entropy of entanglement, then we can make the statement that not all tripartite pure states are GHZ reducible.

V. CONCLUSIONS

In the above discussions, it is shown that the set of generalized GHZ states is not a minimal reversible entanglement generating set, a MREGS for m -party pure states ($m \geq 4$) generally includes states other than the generalized GHZ states, for 4-party pure states, there must be at least 12 member states in a MREGS.

For the GHZ reducible tripartite pure states, there are strong relations among the relative entropies of entanglement. And the additivity of the relative entropy of entanglement is shown to be a necessary condition for all the tripartite pure states to be GHZ reducible.

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VII. APPENDIX A: ANOTHER PROOF OF PROPOSITION 2

Before the proof, we first state another lemma.

Lemma 2: For bipartite quantum state

$$\begin{aligned} \rho &= \sum_{n_1 n_2} a_{n_1 n_2} |\phi_{n_1} \psi_{n_1}\rangle \langle \phi_{n_2} \psi_{n_2}| \\ &\equiv \sum_{n_1 n_2} a_{n_1 n_2} |\phi_{n_1}\rangle_A \langle \phi_{n_2}| \otimes |\psi_{n_1}\rangle_B \langle \psi_{n_2}| \end{aligned} \quad (35)$$

the relative entropy of entanglement is given by

$$E_r(\rho) = - \sum_n a_{nn} \log_2 a_{nn} - S(\rho) \quad (36)$$

where $|\phi_n\rangle$ ($|\psi_n\rangle$) is a set of orthogonal normalized states of system A (B), $S(\rho) \equiv \text{tr}_{AB}(-\rho \log_2 \rho)$ is the von Neumann entropy.

This lemma is a extension of Vedral and Plenio's theorem (Theorem 3 in ref. [18]), the proof is similar to that in ref. [18], details can be found in ref. [21], this lemma can also follow directly from Rains' theorem 9 in ref. [19].

Now we come to the proof of proposition 2. The following pure states are written in their Schmidt decomposition form,

$$\begin{aligned} |\psi\rangle_{AB}^{\otimes m} &= \sum_i \sqrt{p_i^\psi} |i^{A_1}\rangle |i^{B_1}\rangle \\ |\varphi\rangle_{AC}^{\otimes n} &= \sum_i \sqrt{p_i^\varphi} |i^{A_2}\rangle |i^{C_1}\rangle \\ |\phi\rangle_{BC}^{\otimes l} &= \sum_i \sqrt{p_i^\phi} |i^{B_2}\rangle |i^{C_2}\rangle \\ |\Theta\rangle_{ABC}^{\otimes k} &= \sum_i \sqrt{p_i^\Theta} |i^{A_3}\rangle |i^{B_3}\rangle |i^{C_3}\rangle \end{aligned} \quad (37)$$

where p_i^α ($\alpha = \psi, \varphi, \phi, \Theta$) satisfy the normalization condition $\sum_i p_i^\alpha = 1$, the systems A_k (B_k, C_k) ($k = 1, 2, 3$)

are held by Alice (Bob, Claire). Since for pure states the relative entropy of entanglement is additive [18], we have

$$E_r(|\psi\rangle_{AB}^{\otimes m}) = m \cdot E_r(|\psi\rangle_{AB}) = - \sum_i p_i^\psi \log_2 p_i^\psi \quad (38)$$

Set $|\Psi_1\rangle_{ABC} = |\psi\rangle_{AB}^{\otimes m} \otimes |\Theta\rangle_{ABC}^{\otimes k}$, then

$$\begin{aligned} |\Psi_1\rangle_{ABC} &= \sum_i \sqrt{p_i^\psi} |i^{A_1}\rangle |i^{B_1}\rangle \\ &\quad \otimes \sum_j \sqrt{p_j^\Theta} |j^{A_3}\rangle |j^{B_3}\rangle |j^{C_3}\rangle \\ &= \sum_{ij} \sqrt{p_i^\psi p_j^\Theta} |ij\rangle_A \otimes |ij\rangle_B \otimes |j\rangle_C \end{aligned} \quad (39)$$

therefore

$$\begin{aligned} \rho_{AB}^{|\Psi_1\rangle} &\equiv \text{tr}_C \{ |\Psi_1\rangle_{ABC} \langle \Psi_1| \} \\ &= \sum_{ii'j} \sqrt{p_i^\psi p_{i'}^\psi} \cdot p_j^\Theta |ij\rangle_A \langle i'j| \otimes |ij\rangle_B \langle i'j| \end{aligned} \quad (40)$$

From lemma 2, it follows that

$$\begin{aligned} E_r(\rho_{AB}^{|\Psi_1\rangle}) &= - \sum_{ij} p_i^\psi p_j^\Theta \log_2 (p_i^\psi p_j^\Theta) - S(\rho_{AB}^{|\Psi_1\rangle}) \\ &= - \sum_i p_i^\psi \log_2 p_i^\psi - \sum_j p_j^\Theta \log_2 p_j^\Theta \\ &\quad + \sum_j p_j^\Theta \log_2 p_j^\Theta \\ &= - \sum_i p_i^\psi \log_2 p_i^\psi \\ &= m \cdot E_r(|\psi\rangle_{AB}) \end{aligned} \quad (41)$$

We now come to prove that

$$E_r(\rho_{A_1 B_1}) = E_r(\rho_{A_1 B_1} \otimes \rho_{A_2} \otimes \rho_{B_2}) \quad (42)$$

It is known that [18]

$$\begin{aligned} E_r(\rho_{A_1 B_1} \otimes \rho_{A_2} \otimes \rho_{B_2}) &\leq E_r(\rho_{A_1 B_1}) + E_r(\rho_{A_2} \otimes \rho_{B_2}) \\ &= E_r(\rho_{A_1 B_1}) \end{aligned} \quad (43)$$

On the other hand, Alice and Bob can perform local unitary transformations and measurements to transform the state $\rho_{A_1 B_1} \otimes \rho_{A_2} \otimes \rho_{B_2}$ into the state $\rho_{A_1 B_1} \otimes |0\rangle_{A_2} \langle 0| \otimes |0\rangle_{B_2} \langle 0|$, as the relative entropy of entanglement does not increase under LOCC, there is

$$\begin{aligned} E_r(\rho_{A_1 B_1} \otimes \rho_{A_2} \otimes \rho_{B_2}) &\geq E_r(\rho_{A_1 B_1} \otimes |0\rangle_{A_2} \langle 0| \\ &\quad \otimes |0\rangle_{B_2} \langle 0|) \\ &= E_r(\rho_{A_1 B_1}) \end{aligned} \quad (44)$$

The last equality is true since there is no limit on the dimension of the Hilbert space for the systems held by

Alice and Bob. Therefore eq. (42) follows from eq. (43) and (44).

From eq. (41) and (42), we have

$$E_r(A, B) = E_r(\rho_{AB}^{|\Phi\rangle}) = E_r(\rho_{AB}^{|\Psi_1\rangle}) = m \cdot E_r(|\psi\rangle_{AB}) \quad (45)$$

The other two equalities in proposition 2 follow from the symmetry of the state $|\Phi\rangle_{ABC}$. Thus the proof of proposition 2 is completed.

VIII. APPENDIX B: PROOF OF PROPOSITION 4

Let ρ_{AB} and ρ_{BC} be separable states, then

$$\rho_{BC} = \sum_{i=1}^M p_i |\psi_i^B\rangle \langle \psi_i^B| \otimes |\phi_i^C\rangle \langle \phi_i^C| \quad (46)$$

where $\varepsilon = \{p_i, |\psi_i^B\rangle \langle \phi_i^C| | i = 1, 2, \dots, M\}$ is an ensemble of ρ_{BC} with the fewest members.

Let us first show that, *the states $|\phi_i^C\rangle$ in eq. (46) can always be chosen to be orthogonal.*

Alice appends an ancilla and performs a local unitary transformation on $|\Psi\rangle_{ABC}$, resulting in

$$|\tilde{\Psi}\rangle_{ABC} = \sum_{i=1}^M \sqrt{p_i} |i^A\rangle \psi_i^B \phi_i^C \quad (47)$$

where $|i^A\rangle$ is a set of orthogonal normalized states of Alice's enlarged system. The Hughston-Jozsa-Wootters result [20] ensures that this is always possible. The reduced density matrix

$$\begin{aligned} \widetilde{\rho_{AB}} &= \text{tr}_C(|\tilde{\Psi}\rangle_{ABC} \langle \tilde{\Psi}|) \\ &= \sum_{i,j=1}^M \sqrt{p_i p_j} \cdot \langle \phi_j^C | \phi_i^C \rangle \cdot |i^A\rangle \langle j^A| \otimes |\psi_i^B\rangle \langle \psi_j^B| \end{aligned} \quad (48)$$

is also a separable state, since local unitary transformation by Alice does not change the entanglement of the two systems A and B, i.e., $\widetilde{\rho_{AB}}$ can be written as

$$\widetilde{\rho_{AB}} = \sum_k p_k \cdot \rho_k^A \otimes \rho_k^B \quad (49)$$

Let P_A be any projection acting on the Hilbert space of system A. It is obvious that the state

$$\rho_P = (P_A \otimes I_B) \widetilde{\rho_{AB}} (P_A \otimes I_B) \quad (50)$$

is also a separable state (except for a normalization factor). Set $P_A = |m^A\rangle \langle m^A| + |n^A\rangle \langle n^A|$, therefore

$$\begin{aligned} \rho_P &= p_m |m^A\rangle \langle m^A| \otimes |\psi_m^B\rangle \langle \psi_m^B| \\ &\quad + p_n |n^A\rangle \langle n^A| \otimes |\psi_n^B\rangle \langle \psi_n^B| \\ &\quad + \sqrt{p_m p_n} \langle \phi_m^C | \phi_n^C \rangle \cdot |n^A\rangle \langle m^A| \otimes |\psi_n^B\rangle \langle \psi_m^B| \\ &\quad + \sqrt{p_m p_n} \langle \phi_n^C | \phi_m^C \rangle \cdot |m^A\rangle \langle n^A| \otimes |\psi_m^B\rangle \langle \psi_n^B| \end{aligned} \quad (51)$$

Let $|\psi_n\rangle = \alpha |\psi_m\rangle + \beta |\psi_m^\perp\rangle$, where $\beta |\psi_m^\perp\rangle \equiv (|\psi_n\rangle - \langle \psi_m | \psi_n \rangle \cdot |\psi_m\rangle)$ is orthogonal to $|\psi_m\rangle$. Let the states $|\psi_m\rangle$ and $|\psi_m^\perp\rangle$ be the basis vectors for the Hilbert space of system B, the partial transpose of ρ_P is written as

$$(\rho_P)^{T_B} = \begin{pmatrix} p_m & 0 & K\alpha & 0 \\ 0 & 0 & K\beta & 0 \\ K^* \alpha^* & K^* \beta^* & p_n \cdot |\alpha|^2 & p_n \alpha \beta^* \\ 0 & 0 & p_n \alpha^* \beta & p_n \cdot |\beta|^2 \end{pmatrix} \quad (52)$$

where $K \equiv \sqrt{p_n p_m} \langle \phi_n^C | \phi_m^C \rangle$. The separability of ρ_P ensures the positivity of its partial transpose $(\rho_P)^{T_B}$ [22], this positivity requires

$$\langle \phi_n^C | \phi_m^C \rangle = 0 \text{ or } \beta = 0 \quad (53)$$

i.e., for all $i \neq j$, there is

$$|\phi_j^C\rangle \perp |\phi_i^C\rangle \text{ or } |\psi_j^B\rangle = |\psi_i^B\rangle \quad (54)$$

If $|\psi_j^B\rangle \neq |\psi_i^B\rangle$, we have that $|\phi_j^C\rangle \perp |\phi_i^C\rangle$. And if $|\psi_j^B\rangle = |\psi_i^B\rangle \neq |\psi_k^B\rangle$, we can always write

$$\begin{aligned} p_i |\phi_i^C\rangle \langle \phi_i^C| + p_j |\phi_j^C\rangle \langle \phi_j^C| \\ = p'_i |\phi'_i{}^C\rangle \langle \phi'_i{}^C| + p'_j |\phi'_j{}^C\rangle \langle \phi'_j{}^C| \end{aligned} \quad (55)$$

where $p'_i + p'_j = p_i + p_j$ and $|\phi'_i{}^C\rangle \perp |\phi'_j{}^C\rangle$, each of the two states $|\phi'_i{}^C\rangle$ and $|\phi'_j{}^C\rangle$ is a linear addition of the two states $|\phi_i^C\rangle$ and $|\phi_j^C\rangle$, so, $|\phi'_i{}^C\rangle$ and $|\phi'_j{}^C\rangle$ are also orthogonal to $|\psi_k^B\rangle$. That is to say, we can rewrite ρ_{BC} as

$$\rho_{BC} = \sum_{i=1}^M p'_i |\psi_i^B\rangle \langle \psi_i^B| \otimes |\phi'_i{}^C\rangle \langle \phi'_i{}^C| \quad (56)$$

where $|\phi'_i{}^C\rangle$ is a set of orthogonal normalized states of system C. Thus we prove that, *the states $|\phi_i^C\rangle$ in eq. (46) can always be chosen to be orthogonal.*

Then Alice can append an ancilla and perform a local unitary transformation on $|\Psi\rangle_{ABC}$, resulting in

$$|\tilde{\tilde{\Psi}}\rangle_{ABC} = \sum_{i=1}^m \sqrt{p'_i} |i^A\rangle \psi_i^B \phi'_i{}^C \quad (57)$$

where $\sum_i p'_i = 1$ and $|i^A\rangle$ ($|i^C\rangle \equiv |\phi'_i{}^C\rangle$) is a set of orthogonal normalized states of system A (C), while $|\psi_i^B\rangle$ is a set of normalized states of system B, not necessarily orthogonal. We have that

$$\begin{aligned}\rho'_{AC} &= \text{tr}_B \left(\left| \widetilde{\Psi} \right\rangle_{ABC} \left\langle \widetilde{\Psi} \right| \right) \\ &= \sum_{ij} \sqrt{p'_i p'_j} \langle \psi_j^B | \psi_i^B \rangle \cdot |i^A j^C\rangle \langle j^A i^C| \quad (58)\end{aligned}$$

Since local unitary transformation do not change the relative entropy of entanglement as well as the von Neumann entropies, from lemma 2, we have

$$\begin{aligned}E_r(A, C) &= E_r(\rho_{AC}) = E_r(\rho'_{AC}) \\ &= - \sum_i p'_i \log_2 p'_i - S(\rho_{AC}) \quad (59) \\ &= H\{p'_i\} - S(\rho_B)\end{aligned}$$

where $H\{p'_i\} \equiv - \sum_i p'_i \log_2 p'_i$. Since ρ_{AB} , ρ_{BC} are separable states, there is

$$E_r(A, B) = E_r(B, C) = 0 \quad (60)$$

And

$$\begin{aligned}\rho'_A &= \text{tr}_C(\rho'_{AC}) = \sum_i p'_i \cdot |i^A\rangle \langle i^A| \\ \rho'_C &= \text{tr}_A(\rho'_{AC}) = \sum_i p'_i \cdot |i^C\rangle \langle i^C| \\ S(\rho_A) &= S(\rho'_A) = - \sum_i p'_i \log_2 p'_i \equiv H\{p'_i\} \quad (61) \\ S(\rho_A) &= S(\rho_C) = H\{p'_i\}\end{aligned}$$

Finally we get the result

$$\begin{aligned}S(\rho_A) + E_r(B, C) &= S(\rho_B) + E_r(A, C) \\ &= S(\rho_C) + E_r(A, B) = H\{p'_i\} \quad (62)\end{aligned}$$

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